Model theory – 3rd Lecture – Ultrafilters & Ultraproducts

- technique to build new models out of models

IDEA

We have a collection of models $(M_i)_{i\in I}$ on some set of indexes I $\rightsquigarrow$ We take the ultraproduct $\prod_{i\in I} M_i / \mathcal{U}$ ← ultrafilter or $\int M_i \, d\mathcal{U}$ (newer notation)

{the symbol ♥ means that part was not part of the lecture and is therefore not mandatory}

Definition Let I be a non-empty set. A "filter $\mathcal{F}$ on I" is a collection of subsets of I such that

- $\emptyset \notin \mathcal{F}$,
- $I \in \mathcal{F}$,
- if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$,
- if $A \in \mathcal{F}$ and $A \subseteq B \subseteq I$, then $B \in \mathcal{F}$

An "ultrafilter on I" is a $\subseteq$-maximal filter on I

Examples Let I be a non-empty set

- Let $\emptyset \neq B \subseteq I$. The collection $\mathcal{F}_B = \{A \subseteq I \mid A \supseteq B\}$ is a filter. "Principal ultrafilter"

In particular, $\mathcal{F}_I = \{I\}$ is a filter

- Let I be infinite. The collection $\mathcal{F}_{\aleph_0} = \{B \subseteq I \mid I \setminus B \text{ is finite}\}$ is a filter called "cofinite filter on I"

Proposition Let $\mathcal{F}$ be a filter on the non-empty set I. The following propositions are equivalent

(i) $\mathcal{F}$ is an ultrafilter,

(ii) For every set $A \subseteq I$ $A \in \mathcal{F}$ or $(I \setminus A) \in \mathcal{F}$

we will abbreviate this with A^c when it is clear by context which is I.

Proof We start with (i) $\Rightarrow$ (ii). Let $\emptyset \neq A \subseteq I$ and suppose $A \notin \mathcal{F}$ and $A^c \notin \mathcal{F}$. Consider the set

$\mathcal{U} = \{B_1 \cap B_2 \subseteq I \mid \text{there exist } C_1, C_2 \in \mathcal{F} \cup \{A^c\} \text{ such that } B_i \supseteq C_i \text{ for } i = 1, 2\}$

We want to show $\mathcal{U}$ is a filter on I extending $\mathcal{F}$, which contradicts to the maximality of $\mathcal{F}$

(i) Suppose $\mathcal{U} \ni \emptyset$. Then there exist $B_1, B_2 \subseteq I$ such that there exist $C_1, C_2 \in \mathcal{F} \cup \{A^c\}$ such that $B_i \supseteq C_i$ for $i = 1, 2$ and $B_1 \cap B_2 = \emptyset$. Notice this means $C_1 \cap C_2 = \emptyset$, which in turn excludes C_1 and C_2 are both in $\mathcal{F}$ (since $\mathcal{F}$ is a filter) and they are both A^c (A^c would be the empty set, so A would be I, against the hypothesis $A \notin \mathcal{F}$). We are left with $A^c \cap C_1 = \emptyset$ for some $C_1 \in \mathcal{F}$. This means $C_1 \subseteq A$, so $A \in \mathcal{F}$

This is a contradiction, so $\phi \notin \mathcal{U}$

(ii) Clearly $B_1 = B_2 = I$ leads to $\mathcal{U} \ni I$

(iii) Suppose $B, C \in \mathcal{U}$ Then there exist B_1, B_2, B_3, B_4 such that

$$\begin{cases} B_1 \cap B_2 = B \\ B_3 \cap B_4 = C \end{cases} \quad \text{and} \quad B_i \supseteq C_i \in \mathcal{F} \cup \{A^C\} \text{ for } i = 1, \dots, 4$$

Now let $B_{\mathcal{F}}$ be the intersection of the B_i whose C_i is in $\mathcal{F}$ (eventually $B_{\mathcal{F}}$ is I if none of the C_i is in $\mathcal{F}$), and B_A be the intersection of the remaining. Then it is easy to check

$$B_A \cap B_{\mathcal{F}} = B \cap C \quad \text{and} \quad B_A, B_{\mathcal{F}} \in \mathcal{F} \cup \{A^C\}$$

(iv) This instantly follows from the definition of $\mathcal{U}$. $\dashv$

which proves $\mathcal{U}$ is a filter and we conclude as said

<u>For (ii) $\Rightarrow$ (i)</u> By contradiction, again, suppose $\mathcal{F}$ is not maximal, i.e., there exists a filter $\mathcal{G}$ on I properly extending $\mathcal{F}$ Then, there exists $A \in \mathcal{G} \setminus \mathcal{F}$ By property of $\mathcal{F}$ we get

$A^C \in \mathcal{F} \subset \mathcal{G}$,

and we conclude $\phi = A \cap A^C \in \mathcal{G}$ from $A, A^C \in \mathcal{G}$ But then $\mathcal{G}$ is <u>not</u> a filter We proved the maximality of $\mathcal{F}$

The following result proves the existance of ultrafilters:

Lemma Every filter is contained in an ultrafilter

Proof (of Lemma): Let $\mathcal{F}$ be a filter on the non-empty set I

Consider the set

$$\Delta_{\mathcal{F}} = \{ \mathcal{G} \text{ filter on } I \mid \mathcal{G} \supseteq \mathcal{F} \}$$

Clearly it is non-empty since $\mathcal{F} \in \Delta_{\mathcal{F}}$ Our aim is to apply Zorn's Lemma to $(\Delta_{\mathcal{F}}, \subseteq)$ and get a $\subseteq$-maximal filter extending $\mathcal{F}$, which of course will be an ultrafilter

Let $\mathcal{C}$ be a chain in $(\Delta_{\mathcal{F}}, \subseteq)$, i.e a totally ordered subset of $(\Delta_{\mathcal{F}}, \subseteq)$, and suppose $\mathcal{C}$ is non-empty (otherwise we can choose $\mathcal{F}$ as $\subseteq$-upper bound for $\mathcal{C}$) We define

$$\mathcal{H} = \bigcup_{\mathcal{G} \in \mathcal{C}} \mathcal{G}$$

We want to show $\mathcal{H} \in \Delta_{\mathcal{F}}$ Since none of the filters contain ϕ, so does $\mathcal{H}$ Let $\mathcal{G} \in \mathcal{C}$, then $\mathcal{H} \supseteq \mathcal{G} \ni I$, i.e $I \in \mathcal{H}$

Suppose $A, B \in \mathcal{H}$ Then, there exist two filters in $\mathcal{C}$, say $\mathcal{G}_A$ and $\mathcal{G}_B$, such that $A \in \mathcal{G}_A$ and $B \in \mathcal{G}_B$ Since $(\mathcal{C}, \subseteq)$ is totally ordered we can assume wlog that $\mathcal{G}_A \subseteq \mathcal{G}_B$ Then, $\mathcal{G}_B \ni B, A$ and by property (3) of filters $\mathcal{G}_B \ni B \cap A$ Since $\mathcal{H}$ extends $\mathcal{G}_B$, we get $B \cap A \in \mathcal{H}$ Lastly, if $A \in \mathcal{H}$ and $A \subseteq B$, we get $\mathcal{G}_A$ as above Then,

$B \in \mathcal{G}_A$ by (4) and we conclude as above

We established $\mathcal{H} \in \Delta_{\mathcal{Y}}$ and it is trivial to check it is an upper bound for $\mathcal{C}$

By Zorn's Lemma there exists a maximal filter on I extending $\mathcal{Y}$

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We switch the attention to measures now. Particularly, let $\mathcal{F}$ be an ultrafilter on I and define

$$\mu_{\mathcal{F}} \quad \mathcal{P}(I) \to 2, \qquad \mu_{\mathcal{F}}(A) = \begin{cases} 1 & \text{if } A \in \mathcal{F} \\ 0 & \text{else} \end{cases}$$

For every such measure it holds that

<u>Proposition</u> $\mu_{\mathcal{F}}$ is finitely additive, i.e. if $A, B \subseteq I$ are disjoint, then $\mu_{\mathcal{F}}(A \cup B) = \mu_{\mathcal{F}}(A) + \mu_{\mathcal{F}}(B)$, exactly when $\mathcal{F}$ is an ultrafilter

<u>Proof</u> First suppose $\mathcal{F}$ is an ultrafilter and $\mu_{\mathcal{F}}$ is as above. Let A, B be any two disjoint subsets of I. First, consider $A, B \notin \mathcal{F}$. Then, $A \cup B \notin \mathcal{F}$, otherwise, by maximality and (iii) of filter, we get

$$A^c \in \mathcal{F},\ A \cup B \in \mathcal{F} \Rightarrow A^c \cap (A \cup B) = B \in \mathcal{F} \ ⚡$$

(without $A \cap B = \phi$ this is a $\subseteq$, but the argument still works)

Then $\mu_{\mathcal{F}}(A \cup B) = 0 = \mu_{\mathcal{F}}(A) + \mu_{\mathcal{F}}(B)$

Now notice that at most one of A and B can be in $\mathcal{F}$, otherwise their intersection (i.e $\emptyset$) is in $\mathcal{F}$. But then

$$1 = \mu_{\mathcal{F}}(A) + \mu_{\mathcal{F}}(B),$$

and clearly $\mu_{\mathcal{F}}(A \cup B) = 1$ since $A \cup B \in \mathcal{F}$ for (iv) of filter $\lrcorner$

Now assume $\mu : \mathcal{P}(I) \to \{0,1\}$ is additive. We want to show that

$$\mathcal{U}_\mu = \{A \subseteq I \mid \mu(A) = 1\}$$

is an ultrafilter, so that $\mu_{\mathcal{U}_\mu} = \mu$

(i) We have $\mu(\emptyset) = \mu(\emptyset) + \mu(\emptyset)$, since $\emptyset$ is disjoint from $\emptyset$, so that the only possibility is $\mu(\emptyset) = 0$. Therefore $\emptyset \notin \mathcal{U}_\mu$

(ii) $\mu(I) = 1$ by definition of measure (this is also true for $\emptyset$, but it is cool it wasn't necessary). Hence $I \in \mathcal{U}_\mu$

(iii) Let $A, B \in \mathcal{U}_\mu$. Suppose $\mu(A \cap B) = 0$ and let $A' = A \setminus B$ and $B' = B \setminus A$.

Since $(A \cap B) \cap A' = \emptyset = (A \cap B) \cap B'$, we get

$$\mu(A \cap B) + \mu(A') = \mu((A \cap B) \cup A') = \underbrace{\mu(A) = 1}_{\text{since } A, B \in \mathcal{U}_\mu} \text{ so } \mu(A') = 1$$

$$\mu(A \cap B) + \mu(B') = \mu((A \cap B) \cup B') = \mu(B) = 1 \text{ so } \mu(B') = 1$$

But then, since $A' \cap B' = \emptyset$, we get $\mu(A' \cup B') = \mu(A') + \mu(B') = 2$

Absurd. We conclude $\mu(A \cap B) = 1$, so $A \cap B \in \mathcal{U}_\mu$

(iv) Let $A \in \mathcal{U}_\mu$ and $B \supseteq A$. Then $A \cap (B \setminus A) = \emptyset$, so that

$$1 = \mu(A) \leq \mu(A) + \mu(B \setminus A) = \mu(B) \leq 1$$

That is to say $\mu(B) = 1$, so $B \in \mathcal{U}_\mu$

Finally, if $A \notin \mathcal{U}_\mu$, we get

$$1 = \mu(I) = \mu(A \cup A^c) \overset{\text{since } A \cap A^c = \emptyset}{=} \mu(A) + \mu(A^c) \underset{\text{since } A \notin \mathcal{U}_\mu}{=} \mu(A^c)$$

i.e., $A^c \in \mathcal{U}_\mu$. This concludes the proof thanks to Proposition.

$\blacksquare$

We go back to theories. Let Φ be a theory in a language $\mathcal{L}$ and $(\mathcal{M}_i)_{i \in I}$ be a collection of models of Φ indicized by a set of indexes I, then, in general, the product $\prod_{i \in I} \mathcal{M}_i$ is <u>not</u> a model.

For example, we know this happens with fields.

However, we do know it is an $\mathcal{L}$-structure. Let $\mathcal{L}$, $(\mathcal{M}_i)_{i \in I}$, I be as above and let $\mathcal{U}$ be an ultrafilter on I. Consider the equivalence relation on $\prod_{i \in I} \mathcal{M}_i$ given by

$$(a_i) \equiv_{\mathcal{U}} (b_i) \quad \text{iff} \quad \{ i \in I \mid a_i = b_i \} \in \mathcal{U}$$

We are now ready to give the definition of ultraproduct.

<u>Definition</u> With the notation introduced above, the "ultraproduct over the family $(\mathcal{M}_i)_{i \in I}$ with respect to $\mathcal{U}$", in symbols $\int \mathcal{M}_i \, d\mathcal{U}$, is

$$\prod_{i \in I} \mathcal{M}_i \Big/ \equiv_{\mathcal{U}} .$$

The following result justifies such a construction

Theorem (~~Łoś~~ Łoś): A formula φ is true in $\int M_i \, d\mathcal{U}$ if and only if the set of indexes in which it holds $J_\varphi = \{i \in I \mid M_i \models \varphi\}$ lives in $\mathcal{U}$

Proof We proceed by induction on the complexity of φ

Induction basis: If $\varphi \equiv \bot$ the thesis is trivial to prove. Let $\varphi \equiv t_1 = t_2$

We have $\int M_i \, d\mathcal{U} \models \varphi$ iff $(t_{1,i}) = (t_{2,i})$ iff $\{i \in I \mid t_{1,i} = t_{2,i}\} \in \mathcal{U}$ iff $\underbrace{\{i \in I \mid M_i \models \varphi\}}_{= J_\varphi} \in \mathcal{U}$ (Notice $t_{1,i}$ is $t_1^{M_i}$ for every $i \in I$, and conseq $(t_{1,i}), (t_{2,i})$)

Induction step: we distinguish the cases according to main connective:

($\wedge$) Suppose $J_{\psi_1 \wedge \psi_2} \in \mathcal{U}$, then $J_{\psi_1}, J_{\psi_2} \supseteq J_{\psi_1 \wedge \psi_2}$ and they live in $\mathcal{U}$

By IH ψ_1 and ψ_2 are true in $\int M_i d\mathcal{U}$, and so is $\psi_1 \wedge \psi_2$

Walking backwards we just need to add $J_{\psi_1 \wedge \psi_2} = J_{\psi_1} \cap J_{\psi_2}$, which is easy to check for oneself

($\vee$) We observe that if $A \cup B \in \mathcal{U}$, then $A \in \mathcal{U} \vee B \in \mathcal{U}$, otherwise $A^C, B^C \in \mathcal{U}$ and $A^C \cap B^C \in \mathcal{U}$, so $(A^C \cap B^C) \cap (A \cup B) = \emptyset \in \mathcal{U}$

For this reason if $J_{\psi_1 \vee \psi_2} \in \mathcal{U}$, we can assume $J_{\psi_1} \in \mathcal{U}$, then ψ_1 is true in $\int M_i \, d\mathcal{U}$. We can conclude $\psi_1 \vee \psi_2$ is true in $\int M_i \, d\mathcal{U}$

Viceversa, we trace the steps back distinguishing the cases (1) ψ_1 is true in $\int M_i d\mathcal{U}$ and (2) ψ_1 is <u>not</u> true in $\int M_i d\mathcal{U}$

The rest of the proof works similarly ∎

Exercise Prove the compactness theorem from Łoś theorem

Remark If $\mathfrak{M}_i = \mathfrak{M}$ $\forall i \in I$, then $\int_{i \in I} \mathfrak{M}_i \, d\mathcal{U}$ is called "ultrapower"
Definition